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On the New Conditions for a Total Neutrino Conversion in a Medium

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Abstract

We show that the arguments forming the basis for the claim that the conditions for total neutrino conversion derived and studied in detail in [1, 2] "are just the conditions of the parametric resonance of neutrino oscillations supplemented by the requirement that the parametric enhancement be complete", given in [4] have flaws which make the claim physically questionable. We show also that in the case of the transitions in the Earth of the Earth-core-crossing solar and atmospheric neutrinos the peaks in the relevant transitions probabilities P_{ab}, associated with the new conditions, $max\ P_{ab} = 1$, are of physical relevance - in contrast to what is suggested in [4]. Actually, the enhancement of P_{ab} in any region of the corresponding parameter space are essentially determined by these absolute maxima of P_{ab}. We comment on few other aspects of the results derived in [1, 2, 3] which have been misunderstood and/or misinterpreted in [4].

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1. In a *Comment* on our results derived and discussed briefly in [1] and in detail in [2] and on the results derived by one of us in [3], Akhmedov and Smirnov claim [4], in particular, that "the conditions for total neutrino conversion studied by Chizhov and Petcov [1] [2] are just the conditions of the parametric resonance of neutrino oscillations supplemented by the requirement that the parametric enhancement be complete." We show below that the arguments forming the basis for these claims have flaws which, in our opinion, make the claims physically questionable. We show also that in the case of the transitions in the Earth of the Earth-core-crossing solar and atmospheric neutrinos the peaks in the relevant transitions probabilities $P_{\alpha\beta}$, associated with the new conditions, $max\ P_{\alpha\beta} = 1$, are of physical relevance - in contrast to what is suggested in [4]. Actually, the enhancement of $P_{\alpha\beta}$ in any region of the corresponding parameter space is essentially determined by these absolute maxima of $P_{\alpha\beta}$. We comment on few other aspects of the results derived in [1, 2, 3] which have been misunderstood and/or misinterpreted in [4].

The form of the *new conditions* for a total neutrino conversion in a *medium consisting of two or three (nonperiodic) constant density layers*, derived in [1, 2], the region of the parameter space (i.e., the $\Delta m^2/E - \sin^2 2\theta$ plane) where they can be realized, and the physical interpretation of the corresponding absolute maxima of the neutrino transition probabilities of interest as being caused by a *maximal constructive interference* between the amplitudes of the neutrino transitions in the (two) different constant density layers, found in our studies, made us conclude in [1, 2] that these *new conditions* differ from the conditions for parametric resonances in the neutrino transitions, discussed in the articles [5, 6, 7] and possible in a medium with density, varying periodically along the neutrino path. The *Comment* [4] does not provide viable arguments against our conclusions.

2. Consider transitions of neutrinos crossing a medium consisting of i) two layers with different constant densities $N_{1,2}$ and widths $L_{1,2}$, or of ii) three layers of constant density, with the first and the third layers having identical densities N_1 and widths L_1, which differ from those of the second layer, N_2 and L_2 [3, 1, 2]. Suppose the transitions are caused by two-neutrino mixing in vacuum with mixing angle θ. Let us denote by θ_i and $2\phi_i$, $i = 1, 2$, the mixing angle in matter in the layer with density N_i and the phase difference acquired by the two neutrino energy-eigenstates after neutrinos have crossed this layer. It proves convenient to introduce the quantities (see, e.g., [2]):

$$\cos\Phi \equiv Y = c_1 c_2 - \cos(2\theta_2 - 2\theta_1) s_1 s_2, \tag{1}$$

$$\mathbf{X}^2 = 1 - Y^2, \tag{2}$$

$$n_3 \sin\Phi \equiv X_3 = -(s_1 c_2 \cos 2\theta_1 + c_1 s_2 \cos 2\theta_2), \tag{3}$$

[2] By using the verb "studied" in connection with our new conditions for a total neutrino conversion in a medium [1, 2], the authors of [4] suggest indirectly that these conditions were already considered in the literature before the publication of our papers. We would like to note that none of our *new conditions for a total neutrino conversion in a medium* were derived, postulated and/or discussed in some form in the literature on the subject published before the articles [1, 2].

X_3 being the third component of the vector [3] $\mathbf{X} = (X_1, X_2, X_3)$, whose first two components are also given in terms of θ_i and ϕ_i (see [2]). The probability of the transition $\nu_a \to \nu_b$ (i.e., $\nu_e \to \nu_{\mu(\tau)}$, $\nu_\mu \to \nu_e$, etc.) after neutrinos have crossed n alternating layers with densities N_1 and N_2 is given according to [4] by:

$$P(\nu_a \to \nu_b;\ nL) = \left(1 - \frac{X_3^2}{\mathbf{X}^2}\right) \sin^2 \Phi_p \equiv \frac{X_1^2 + X_2^2}{X_1^2 + X_2^2 + X_3^2} \sin^2 \Phi_p, \tag{4}$$

where $\Phi_p = (n/2)\Phi$ if the number of layers n is even, and

$$\Phi_p = \frac{n-1}{2}\Phi + \varphi \ , \quad \varphi = \arcsin\left(s_1 \sin 2\theta_1 / \sqrt{1 - X_3^2/|\mathbf{X}|^2}\right) \tag{5}$$

for odd number of layers with the first layer having density N_1.

The first thing to note is that the expression for φ in eq. (5), given in [4], is strictly speaking incorrect: it is valid only if $Z \geq 0$, where

$$Z = s_2 \sin 2\theta_2 + s_1 (\sin 2\theta_1) Y. \tag{6}$$

The correct expression for φ for arbitrary *sign* Z reads:

$$\varphi = \arctan\left(s_1 \sin(2\theta_1)\ |\mathbf{X}|/Z\right) \ . \tag{7}$$

The authors of [4] demonstrate the same imprecision in eq. (2) of their *Comment*: the functions $\arccos Y$ and $\arcsin |\mathbf{X}|$ have different defining regions and it is incorrect to write $\arccos Y = \arcsin |\mathbf{X}|$: this equality is obviously wrong when $Y < 0$.

According to the authors of [4], "Eqs. (4) and (5) describe the parametric oscillations with the pre-sine factor in (4) and Φ_p being the oscillation depth and phase." and further "Parametric resonance occurs when the depth of the oscillations becomes equal to unity. The resonance condition is therefore $X_3 = 0$."

In the two-layer case i) considered by us in [1, 2], which is relevant for the present discussion, one has $\Phi_p = \Phi$ ($n = 2$). This result and eqs. (1) and (2) imply that actually $\sin^2 \Phi_p = X_1^2 + X_2^2 + X_3^2$. Correspondingly, the "parametric-resonance" form in which the authors of [4] cast the probability $P(\nu_a \to \nu_b;\ nL)$ in this case is artificial: the probability is given by [2]

$$P(\nu_a \to \nu_b;\ 2L) = X_1^2 + X_2^2 = 1 - Y^2 - X_3^2, \tag{8}$$

where we have used eq. (2). Therefore any resonance interpretation of the probability $P(\nu_a \to \nu_b;\ 2L)$ based solely on the eq. (4) with $n = 2$ seems to us physically unjustified. The *new conditions for a total neutrino conversion in a medium* follow in this case from the form (8) of $P(\nu_a \to \nu_b;\ 2L)$ and read [1, 2]:

$$Y = 0, \qquad X_3 = 0. \tag{9}$$

[3] To facilitate the understanding of the main points of our criticism of [4], we use the same notations as in [4] for most of the quantities discussed.

It should be clear from eqs. (8) and (9) that the condition $X_3 = 0$ *alone does not ensure the existence even of a local maximum of the probability* $P(\nu_a \to \nu_b;\ 2L)$.

It is not guaranteed *a priori* that the equations in (9) have non-trivial solutions in general, and in the specific case of transitions of neutrinos crossing the Earth core and the Earth mantle on the way to the detector. As we have shown in [1, 2], the solutions of the conditions (9) i) exist and are given by

$$\textit{solution } A^{(2)}: \quad \begin{cases} \tan\phi_1 = \pm\sqrt{\dfrac{-\cos(2\theta_2)}{\cos(2\theta_1)\cos(2\theta_2 - 2\theta_1)}}\,, \\ \tan\phi_2 = \pm\sqrt{\dfrac{-\cos(2\theta_1)}{\cos(2\theta_2)\cos(2\theta_2 - 2\theta_1)}}\,, \end{cases} \tag{10}$$

where the signs are correlated, and ii) they can be realized *only in the region determined by the three inequalities*:

$$\textit{region } A^{(2)}: \quad \begin{cases} \cos 2\theta_1 \geq 0 \\ \cos 2\theta_2 \leq 0 \\ \cos(2\theta_2 - 2\theta_1) \geq 0. \end{cases} \tag{11}$$

It was demonstrated in [1, 2] as well that the two conditions in eq. (9), or equivalently the solutions expressed by eqs. (10) and (11), are conditions for a *maximal constructive interference* between the amplitudes of the neutrino transitions in the two layers. Thus, a natural physical interpretation of the absolute maxima of $P(\nu_a \to \nu_b;\ 2L)$ associated with the conditions (9) is that of *constructive interference maxima*.

It should be clear from the above arguments that we do not see physical reasons to call $X_3 = 0$ in the case under discussion a "parametric resonance condition". Using the trick of [4] one can easily cast the probability of two-neutrino oscillations in vacuum and in matter with constant density, for example, in the "parametric-resonance" form (4). The analog of the condition $X_3 = 0$ reduces in these cases respectively to $\sin^2 2\theta = 1$ and $\sin^2 2\theta_m = 1$, θ_m being the mixing angle in matter. Thus, according to the terminology suggested in [4], one should call the condition $\sin^2 2\theta = 1$ and the MSW resonance condition $\sin^2 2\theta_m = 1$ "parametric-resonance conditions". One can use such a terminology, of course, but this is not justified by the physics of the process and nobody uses it. The situation in the two-layer case discussed above is in essence the same.

Analogous results and conclusions are valid in the case of medium with three layers (case ii)) considered in [3, 1, 2] ($n = 3$ in eqs. (4) and (5)). Using the correct expression for φ, eq. (7), one finds again that the "parametric resonance" form in which the authors of [4] write the probability of interest $P(\nu_a \to \nu_b;\ 3L)$ is artificial: the probability has the same form as in eq. (8) [2]

$$P(\nu_a \to \nu_b;\ 3L) = 1 - \bar{Y}^2 - \bar{X}_3^2, \tag{12}$$

where

$$\bar{Y} = -c_2 + 2c_1 Y, \tag{13}$$

and

$$\bar{X}_3 = -s_2\cos 2\theta_2 - 2s_1\cos(2\theta_1)\ Y. \tag{14}$$

The conditions for a total neutrino conversion in this case read [2]:

$$\bar{Y} = 0, \qquad \bar{X}_3 = 0. \tag{15}$$

Their solutions in the case of the inequality $N_1 < N_2$ (corresponding to the relation between the densities in the Earth mantle and core) were given in [1, 2] and have the form:

$$solution\ A^{(3)} : \begin{cases} \tan\phi_1 = \pm\sqrt{\dfrac{-\cos 2\theta_2}{\cos(2\theta_2 - 4\theta_1)}}, \\ \tan\phi_2 = \pm\dfrac{\cos 2\theta_1}{\sqrt{-\cos(2\theta_2)\cos(2\theta_2 - 4\theta_1)}}, \end{cases} \tag{16}$$

where the signs are again correlated. The solutions can only be realized in the region

$$region\ A^{(3)} : \begin{cases} \cos(2\theta_2) \le 0, \\ \cos(2\theta_2 - 4\theta_1) \ge 0. \end{cases} \tag{17}$$

It is easy to show that for any number of layers n, the denominator in (4) is always canceled by $\sin^2\Phi_p$ and $P(\nu_a \to \nu_b;\ nL)$ is just a polynomial without any resonance-like feature. Indeed, for even n, as can be shown, we have

$$\sin\Phi_p = \sin\Phi\ U_{n/2-1}(\cos\Phi), \tag{18}$$

where $U_n(x)$ is well-known Chebyshev's polynomial of the second kind [8]. In the case of odd number of layers ($n \ge 3$) one finds

$$P(\nu_a \to \nu_b;\ nL) = \left[s_1 \sin 2\theta_1 \cos\left(\frac{n-1}{2}\Phi\right) + Z U_{\frac{n-3}{2}}(\cos\Phi)\right]^2. \tag{19}$$

Finally, similar considerations apply to the probability of the $\nu_2 \to \nu_e$ transitions, ν_2 being the heavier of the two vacuum mass-eigenstate neutrinos, in the three-layer case ii), $P(\nu_2 \to \nu_e;\ 3L)$. This probability can be used to account for the the Earth matter effects in the transitions of solar neutrinos traversing the Earth: $P(\nu_2 \to \nu_e;\ 3L)$ corresponds [3] to the case of solar neutrinos crossing the Earth mantle, the core and the mantle again on the way to the detector. As was shown in [2], the conditions for a total $\nu_2 \to \nu_e$ conversion, *max* $P(\nu_2 \to \nu_e;\ 3L) = 1$, read:

$$\bar{Y} = 0, \qquad \bar{X}'_3 = 0, \tag{20}$$

where $\bar{Y}$ is given by eq. (13) and

$$\bar{X}'_3 = -s_2 \cos(2\theta_2 - \theta) - 2 s_1 \cos(2\theta_1 - \theta)\ Y. \tag{21}$$

The solutions of the conditions (20) providing the absolute maxima of $P(\nu_2 \to \nu_e;\ 3L)$ and the region where these solutions can take place were given in [1, 2]; they can formally be obtained from eqs. (16) and (17) by replacing $2\theta_1$ and $2\theta_2$ with $(2\theta_1 - \theta)$ and $(2\theta_2 - \theta)$.

The three sets of two conditions, eqs. (9), (15) and (20), and/or their solutions (e.g., eqs. (10) and (16)), and/or the regions where the solutions can be realized (e.g.,

eqs. (11) and (17)), were not derived and/or discussed in any form in [5, 6, 7] or in any other article on the subject of neutrino transitions in a medium published before [1, 2]. None of them follows from the conditions of enhancement of $P(\nu_a \rightarrow \nu_b)$ found in [5, 6, 7] and thus they not a particular case of the latter. That is the reason we used the term "new conditions for a total neutrino conversion in a medium" for them.

The authors of [4] write further: "One well known realization of the parametric resonance condition", i.e., of $X_3 = 0$, is [5, 6, 7, 3] [4] "$c_1 = c_2 = 0$, or

$$2\phi_1 = \pi + 2\pi k_1, \quad 2\phi_2 = \pi + 2\pi k_2, \; k_1, k_2 = 0, 1, 2, ..., \tag{22}$$

independently of the mixing angles." Contrary to what the authors of [4] claim, the two conditions in eq. (22) were not given in the articles [5, 6, 7]: what one finds in these articles *at most* is the condition $2\phi_1 + 2\phi_2 = 2\pi + 2\pi k$ which is not equivalent to the two conditions in eq. (22). The two conditions in eq. (22) were discussed in detail for the 3-layer case in [3]. Moreover, as we have shown in [2] and would like to emphasize here again, *the conditions $c_1 = 0$, $c_2 = 0$ by themself do not lead to a maximum of the neutrino transition probabilities of interest in the neutrino energy variable, unless a third nontrivial condition is fulfilled.* This third condition has the following form for the $\nu_a \rightarrow \nu_b$ (i.e., $\nu_e \rightarrow \nu_{\mu(\tau)}$, $\nu_\mu \rightarrow \nu_e$, etc.) transitions in the two-layer (n = 2) and three-layer (n=3) medium cases i) and ii), respectively [2]: $\cos(2\theta_2 - 2\theta_1) = 0$ and $\cos(2\theta_2 - 4\theta_1) = 0$. For the probability $P(\nu_2 \rightarrow \nu_e;\ 3L)$ it reads [2]: $\cos(2\theta_2 - 4\theta_1 + \theta) = 0$. It is not difficult to convince oneself that the indicated sets of *three* conditions represent possible solutions respectively of (9), (15) and (20) [2]. More generally, the condition $X_3 = 0$ *alone does not guarantee the existence even of a local maximum of the neutrino transition probabilities of interest.*

In what regards the article by Q.Y. Liu and A. Yu. Smirnov quoted in [4] in connection with the conditions (22) (see ref. [5] in [4]), these authors noticed that in the case of muon neutrinos crossing the Earth along the specific trajectory characterized by a Nadir angle $h \cong 28.4°$, and for $\sin^2 2\theta \cong 1$ and $\Delta m^2/E \cong (1{-}2)\times 10^{-4}$ eV^2/GeV, the $\nu_\mu \rightarrow \nu_s$ transition probability, ν_s being a sterile neutrino, is enhanced. The authors interpreted this enhancement as being due to the conditions $2\phi_j = \pi$, $j = 1, 2$, which they claimed to be approximately satisfied. Actually, for the values of the parameters of the examples chosen by Q.Y. Liu and A. Yu. Smirnov to illustrate their conclusion one has $2\phi_1 \cong (0.6 - 0.9)\pi$ and $2\phi_2 \cong (1.2 - 1.5)\pi$. The indicated enhancement is due to [2] the existence of a *nearby total neutrino conversion point* which for $h \cong 28.4°$ is located at $\sin^2 2\theta \cong 0.94$ and $\Delta m^2/E \cong 2.4 \times 10^{-4}$ eV^2/GeV and at which $2\phi_1 \cong 0.9\pi$ and $2\phi_2 \cong 1.1\pi$. We have also found in [2] that for each given $h \lesssim 30°$ there are several total neutrino conversion points at large values of $\sin^2 2\theta$, at which the phases $2\phi_1$ and $2\phi_2$ are not necessarily equal to π or to odd multiples of π (see Table 3 in [2]). Thus, the explanation of the enhancement offered by the indicated authors is at best qualitative and incorrect in essence.

That $X_3 = 0$ *alone* is a condition for *local maxima* of the probability $P(\nu_a \rightarrow \nu_b;\ nL)$ was suggested in E. Kh. Akhmedov, Nucl. Phys. B538 (1999) 25 (to be quoted further as NP B538, 25), on the basis of the form of the probability in

[4] We quote here only the references which, in our opinion, are relevant for the present discussion.

eq. (4). However, as we have already emphasized, the condition $X_3 = 0$ alone *does not guarantee the existence even of a local maximum of the neutrino transition probabilities* $P(\nu_a \to \nu_b;\ 2L)$ *and* $P(\nu_a \to \nu_b;\ 3L)$. This should be clear from the "natural" expressions for the probabilities $P(\nu_a \to \nu_b;\ 2L)$ and $P(\nu_a \to \nu_b;\ 3L)$, given by eqs. (8) and (12). Of the three solutions for the extrema of $P(\nu_a \to \nu_b;\ 3L)$ found in NP B538, 25, the solution $c_1 = 0,\ c_2 = 0$ was already discussed in detail in ref. [3] (compare eqs. (11) - (16) and (24) in [3] with conditions (1) on page 37 of NP B538, 25), while the other two correspond to MSW transitions [5] (they were also briefly discussed in [3]).

4. The authors of [4] claim that "The existence of strong enhancement peaks in transition probability P rather than the condition P=1 is of physical relevance.", although they do not give an example of a relevant strong enhancement peak (i.e., local maximum). As we have shown in [2] (see also [3]), the solutions given by eqs. (10) and (16) and those for the probability $P(\nu_2 \to \nu_e;\ 3L)$ are realized in the transitions in the Earth of the Earth-core-crossing neutrinos (solar, atmospheric, accelerator) and lead to observable effects in these transitions. From the extensive numerical studies we have performed in the realistic cases of transitions of neutrinos in the Earth (e.g., neutrinos crossing the Earth core on the way to the detector) *we do not have any evidence about the presence of significant local maxima in the neutrino transition probabilities of interest not related to the peaks of total neutrino conversion.* Actually, our studies show that only the peaks of total neutrino conversion are dominating in $P(\nu_a \to \nu_b;\ 2L)$, $P(\nu_a \to \nu_b;\ 3L)$ and $P(\nu_2 \to \nu_e;\ 3L)$, and correspondingly determine the regions where these probabilities can be significant in the corresponding space of parameters. The peaks considered, e.g., in [3] (see figs. 1 and 2) are points on the "ridges" formed by local maxima, e.g., in the energy variable at fixed values of the other parameters, leading to the peaks of total neutrino conversion, discovered by us (see figs. 6 - 9 and Tables 5 - 6 in [2]). As one can convince oneself using Figs. 6 - 9 from [2], all maxima in Figs. 4, 5, 10 - 13 and 15 in [9], including the relatively small local ones, are related to (and determined by) the presence of corresponding points (peaks) of total neutrino conversion.

5. The authors of [5, 7] studied the effects of *small* density perturbations on the neutrino oscillations, while we have investigated in [3, 1, 2] the different physical problem of *large* "perturbations" of density. In [6] neutrino oscillations in a medium consisting of *even* number n of alternating layers with densities N_1 and N_2 have been considered. However, i) the two layers were assumed to have equal widths, $L_1 = L_2$, and i) an enhancement of the neutrino transitions was found to take place for small vacuum mixing angles at densities $N_{1,2}$ much smaller than the MSW resonance density. The authors of [5, 6] were interested in and found the conditions for the classical parametric resonance in neutrino oscillations, which can take place after many periods of density modulation of the oscillations. From reading the articles [5, 6, 7] it becomes

[5]Let us note that in what regards the cases of neutrino transitions studied in [3, 1, 2], the article NP B538, 25, contains a rather large number of incorrect statements and conclusions. Most of these statements are concentrated in Section 4 of NP B538, 25, where the author discusses the realistic case of transitions of neutrinos crossing the Earth core we were primarily interested in in [3, 1, 2].

clear that their authors had in mind astrophysical applications of their results [6], and the results obtained in [5, 6, 7] may indeed have astrophysical applications. Our studies [3, 1, 2] were concerned primarily with the neutrino oscillations in the Earth. As a consequence, the conditions of the enhancement of $P(\nu_a \to \nu_b)$ obtained in [5, 6, 7] differ from those found by us in [3, 1, 2]. In the two- and three-layer medium cases i) and ii) discussed by us (e.g., in the case of the Earth) the enhancement found in [6], for example, is not realized.

To summarize, after studying ref. [4] and the references quoted therein one can conclude that the *new conditions for a total neutrino conversion in a medium* found and studied in [1, 2] (i.e., the three sets of two conditions, eqs. (9), (15) and (20), and/or their solutions (e.g., eqs. (10) and (16)), and/or and the regions where the solutions can be realized, eqs. (11) and (17)), were indeed new: they were not derived and/or discussed in any form in [5, 6, 7] or in any other article on the subject of neutrino transitions in a medium published before [1, 2]. None of them follows from the conditions of enhancement of $P(\nu_a \to \nu_b)$ obtained in [5, 6, 7] and thus they are not a particular case of the latter. In [4], in particular, the derivation of these conditions, first given in [2], is reproduced. Most importantly, the *new conditions for a total neutrino conversion* were shown in [2] (see also [3]) to be realized in the transitions in the Earth of the Earth-core-crossing neutrinos (solar, atmospheric, accelerator) and to lead to observable effects in these transitions - contrary to the claims made in [4]. As for the physical interpretation of the associated *new effect of total neutrino conversion* in the cases of two-layer and three-layer medium we have considered, we have proven that this is a maximal constructive interference effect. The interpretation of the *effect* based on the expression (4) for the neutrino transitions probabilities, offered in [4], as we have pointed out, is not convincing: expression (4), in particular, has an artificial resonance-like form. For the studies of the *new effect of total neutrino conversion* in the Earth, eqs. (8) and (12) represent one of the several possible natural expressions for the relevant neutrino transition probabilities. The rest is terminology.

[6]In all the concrete examples and corresponding plots in [7], for instance, the physical quantities having a dimension of length are given in units of the solar radius and are comparable with it; the values of the densities used in the examples differ substantially from those met in the Earth - they are noticeably smaller than the Earth mantle and core densities.

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